The Open University

*Mathematics: A Second Level Course*

*Linear Mathematics M201*
*Bridging Material 2*

# PARTIAL DIFFERENTIATION

*Prepared by the Course Team*

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## Contents

## Set Books

D. L. Kreider, R. G. Kuller, D. R. Ostberg and F. W. Perkins, *An Introduction to Linear Analysis* (Addison-Wesley, 1966).
E. D. Nering, *Linear Algebra and Matrix Theory* (John Wiley, 1970).

It is essential to have these books; the course is based on them and will not make sense without them.

## Conventions

Before working through this correspondence text make sure you have read *An Introduction to the Bridging Material* and *A Guide to the Linear Mathematics Course.*

The set books are referred to as:

**K** for *An Introduction to Linear Analysis*
N for *Linear Algebra and Matrix Theory*

## Note

This bridging material is not based on the set books. It has been written especially for the benefit of students who have taken the Mathematics Foundation Course M101 (The Open University Press, 1978)
References to this foundation course take the form M101 Block V *Unit 2*

# 2.1 TAYLOR POLYNOMIALS AND APPROXIMATIONS TO FUNCTIONS

## 2.1.1 Introduction

You will remember from M101 Block III *Unit 4* p18 that the $n^{\text{th}}$ Taylor polynomial for a function $f$, provided $f$ can be differentiated $n$ times, is

$$f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+\cdots+\frac{1}{n!}f^{(n)}(0)x^n$$

You will also have seen in TV14 how to use the Taylor polynomials for the sine function to calculate approximate values of $\sin x$: computer graphics demonstrated that the fifth Taylor polynomial for sine

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}$$

does not differ very much from sine $x$ over most of the interval $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$. Because of the application of Taylor polynomials calculating approximate values of functions, the $n^{\text{th}}$ Taylor polynomial of a function $f$ is called the $n^{\text{th}}$ Taylor approximation to $f$ in M201 and elsewhere. (You may find it worth while to watch M101 TV14 again as you will need this work before reading M201 *Units 19* and *21*).

The important information that we must have before we can use Taylor polynomials to calculate (approximate) values of sine, or of other functions, and have any confidence in our answer, is to know *how accurately* we can expect a Taylor polynomial to approximate the function. The difficulty is precisely the same as the one you met in M101 Block II *Unit 3* in discussing error bounds: in practice, if we are using the Taylor polynomial to calculate an approximate value for a function such as $\sin x$; all we know is the *approximate* value, not the true value, so we don't know the size of the error! The way out of this difficulty is given by Taylor's theorem, which tells us how to calculate our upper limit, or *upper bound* for the error. This upper bound will usually over-estimate the actual error, but it will never the less be very useful.

## 2.1.2 Taylor's Theorem (Simple Case)

The first Taylor approximation (first Taylor polynomial) of a function $f$ is $f(0)+xf'(0)$. Compare this with the second approximation $f(0)+xf'(0)+\frac{1}{2}x^2f''(0)$. The difference is $\frac{1}{2}x^2f''(0)$ and this suggests that an approximation to the error in the value given by the first formula is $\frac{1}{2}x^2f''(0)$. Obviously this can't be the upper bound for the error that we are seeking, but in fact Taylor's theorem tells us that the upper bound looks rather like it. The precise statement is as follows.

Suppose $f$ can be differentiated twice. Then

$$f(x)=f(0)+xf'(0)+C_1(x)$$

where the correction term $C_1(x)$ satisfies

$$|C_1(x)|\le\tfrac{1}{2}Bx^2$$

provided $|f''(t)|\le B$ for all $t\in[0,x]$ (or $t\in[x,0]$ if $x<0$) so if we replace $f''(0)$ in the quadratic term $\frac{1}{2}x^2f''(0)$ by a number $B$ which is bigger than the value of $f''(t)$ over the whole interval $[0,x]$, we get a bound for the error in the first Taylor approximation.

We shall not prove this result here: you will find a proof of the general case on page K667 if you are interested, but a knowledge of the proof is not required for M201.

*Example 1*

To illustrate Taylor's theorem, let us apply it to the sine function. In this case, the first Taylor approximation is

$$\sin x \simeq \sin 0 + x \sin'(0)$$
$$= 0 + x$$

and Taylor's theorem tells us that the error $-C_1(x)$ satisfies

$$|C_1(x)| \le \tfrac{1}{2}Bx^2$$

provided $|\sin''(t) < B$ for all $t \in [0, x]$.

Since we know $\sin''(t) = -\sin t$ and $|\sin t| \le 1$ for all $t$ we can take $B = 1$. Thus Taylor's theorem tells us that

$$|\sin x - x| \le \tfrac{1}{2}x^2$$

For example, if $x = \frac{1}{10}$, we can deduce $\sin \frac{1}{10} = 0.100$ with error at most

$$\tfrac{1}{2}(\tfrac{1}{10})^2 \simeq 0.005$$

Notice that the true value of $\sin \dfrac{\pi}{10}$ to five decimal places is 0.09983 so that the actual error is only about one twentieth of that guaranteed by Taylor's theorem. However, we do have for certain a maximum value for the error, and we got it for very little work. Indeed, as we shall see shortly, this error estimate could have been improved considerably with only a little more care.

*Example 2*

Show that the error in taking $\tan \dfrac{\pi}{6} = \dfrac{\pi}{6} \simeq 0.52$ (the value of the first Taylor approximation to $\tan x$) is less than

$$\frac{\pi^2}{27\sqrt{3}} \simeq 0.21$$

If $\quad f(x) = \tan x,$

then $\quad f'(x) = \sec^2 x$

$\quad\quad f''(x) = 2\sec^2 x \tan x$

so the first Taylor approximation to $\tan x$ is $0 + xf'(0) = x$.

The error for $x = \dfrac{\pi}{6}$ is at most $\left|\dfrac{1}{2}B \cdot \dfrac{\pi^2}{6^2}\right|$ provided $|f''(t)| \le B$ for $t \in \left[0, \dfrac{\pi}{6}\right]$.

Since $\sec t$ and $\tan t$ both increase as $t$ increases on $\left[0, \dfrac{\pi}{6}\right]$ the largest value that $f''(t)$ can take on the interval $\left[0, \dfrac{\pi}{6}\right]$ is

$$2\sec^2 \frac{\pi}{6} \tan \frac{\pi}{6} = 2 \times \left(\frac{2}{\sqrt{3}}\right)^2 \times \frac{1}{\sqrt{3}} = \frac{8}{3\sqrt{3}}.$$

Hence the error is at most $\dfrac{1}{2} \cdot \dfrac{8}{3\sqrt{3}} \cdot \dfrac{\pi^2}{6^2} = \dfrac{\pi^2}{27\sqrt{3}} \simeq 0.21.$

### 2.1.3 Taylor's Theorem (General Case)

The simple case of Taylor's theorem can be generalized in two ways. First, we can obtain an upper bound for the error in using the $n^{\text{th}}$ Taylor approximation.

**Theorem**

Suppose $f$ is differentiable $(n + 1)$ times. Then

$$f(x) = f(0) + xf'(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + C_n(x)$$

where the correction term $C_n(x)$ satisfies

$$|C_n(x)| \leq \left| \frac{x^{n+1}}{(n+1)!} B \right|$$

provided $|f^{(n+1)}(t)| \leq B$ for all $t \in [0, x]$ (or $t \in [x, 0]$ if $x < 0$).

*Example 3*

Use Taylor's theorem to find an upper bound for the error in using the second Taylor approximation to calculate sin 0.1.

*Solution*

Since $\sin'(0) = \cos 0 = 1,$
$\sin''(0) = -\sin 0 = 0$

The second Taylor approximation to sine is the same as the first namely $\sin x \simeq x$.

We can write $\sin x = x + C_2(x)$ where by Taylor's theorem $C_2(x)$ satisfies

$$|C_2(x)| \leq \left| \frac{x^3}{3!} B \right|$$

provided $|\sin'''(t)| \leq B$ for all $t \in [0, x]$.

Now $\sin'''(t) = -\cos t$ and $|\cos t| \leq 1$ for all $t$, so

$$|C_2(x)| \leq \left| \frac{x^3}{3!} \right|.$$

In particular, $\sin 0.1 = 0.1 + C_2(0.1)$

where $|C_2(0.1)| \leq \dfrac{(0.1)^3}{6} \simeq 0.00017$

This answer shows a considerable improvement on the error bound obtained in Example 1, although the approximate value of sin (0.1) is the same in each case.

The other way in which we can generalize Taylor's theorem is so that we can use it not only when we can easily work out $f$ and its derivatives at $x = 0$, but also where $x = \alpha$ is the point where we know something about the function. This is just like the situation in M101 Block III Section 4.4 where we considered Taylor series about a general point. The result that we obtain is

$$f(x) = f(\alpha) + f'(\alpha)(x - \alpha) + \frac{1}{2!} f''(\alpha)(x - \alpha)^2 + \frac{1}{3!} f'''(\alpha)(x - \alpha)^3 + \cdots + \frac{1}{n!} f^{(n)}(\alpha)(x - \alpha)^n + \cdots;$$

### 2.1.4 Taylor's Theorem

If $f$ is differentiable $(n + 1)$ times then

$$f(x) = f(\alpha) + (x - \alpha)f'(\alpha) + \cdots + \frac{(x - \alpha)^n}{n!} f^{(n)}(\alpha) + C_n(x)$$

where $C_n(x)$ satisfies

$$|C_n(x)| \leq \left| \frac{(x - \alpha)^{n+1}}{(n + 1)!} B \right|$$

provided $|f^{(n+1)}(t)| \leq B$ for all $t \in [\alpha, x]$ (or $[x, \alpha]$ if $x < \alpha$).

*Example 4*

Use the second Taylor approximation at $x = 1$ to calculate $\log_e 1.1$. Find an upper bound for the error, using Taylor's theorem.

*Solution*

If $\quad f(x) = \log_e x$

then $\quad f'(x) = x^{-1}$

$$f''(x) = -x^{-2}$$

$$f'''(x) = 2x^{-3}$$

The second Taylor approximation at $x = 1$ is

$$\begin{aligned} f(x) &\simeq f(1) + (x - 1)f'(1) + \tfrac{1}{2}(x - 1)^2 f''(1) \\ &= 0 + (x - 1) - \tfrac{1}{2}(x - 1)^2 \end{aligned}$$

so $\quad f(1.1) \simeq 0.1 - \frac{1}{2}(0.1)^2 = 0.9950.$

By Taylor's theorem, the correction term $C_1(x)$ satisfies

$$|C_1(x)| < |\tfrac{1}{6}(x - 1)^3 B|$$

provided $|f'''(t)| \leq B$ for all $t \in [1, x]$.

Here, $x = 1.1$ and $f'''(t) = 2t^{-3} < 2$ for $t \in [1, 1.1]$.

Hence the error in the estimate $\log_e 1.1 = 0.095$ is less than $\frac{1}{6}(0.1)^3.2 < 0.0004$.

You can now read M201 *Unit 19*, Section 3.4 on Taylor Approximation.

## 2.2 PARTIAL DIFFERENTIATION

### 2.2.1 Introduction

In the case of a function of a single variable, we are able to measure the rate of change of the value of the function relative to a change in the value of the variable by finding the derivative of the function. In the case of a function of two variables, we can seek similarly to measure the rate of change of the value of the function relative to a change in the value of either of the variables. The new concept we need is that of a *partial derivative*.

It is easy to get an intuitive grasp of the notation of a partial derivative. Imagine yourself standing on a hillside at a point where two roads cross, one road running east–west, the other north–south. One road may perhaps have a steep gradient, the other less so. The surface of the hill represents the graph of a function of two variables, Northings and Eastings. (Height depends upon both the latitude and the longitude of the point in question). Speaking roughly, the two partial derivatives of the function 'height' at the crossroads are measured by the slopes of the two roads. If the crossroads happened to be at the top of the hill then each of the slopes would be zero at that point, but elsewhere they will differ.

We want to make this intuitive idea more precise. The geometric example which follows shows the way we must go and helps you to understand the subsequent definitions. But, if you find it hard to visualize three-dimensional figures, do not spend a lot of time trying to understand the example.

*Example*

Consider the surface representing the function

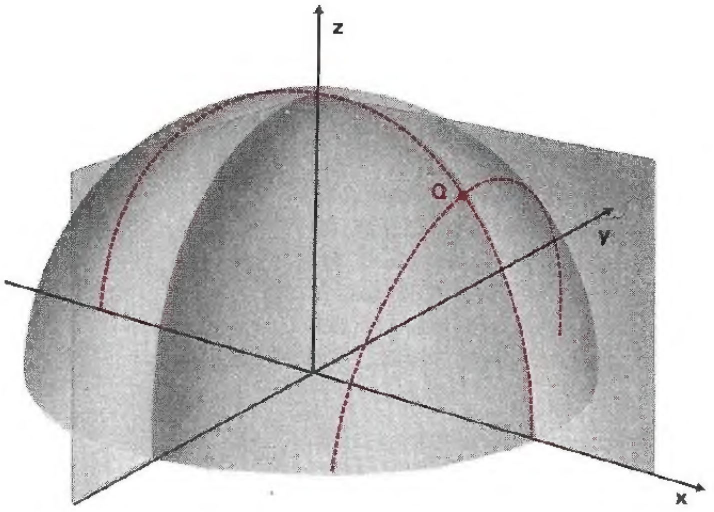
$$z = F(x, y) = \sqrt{1 - (x^2 + y^2)}$$
$$((x, y) \in E \times R, x^2 + y^2 \leq 1)$$

We always take the positive square-root. Then, if we consider $z$ to be 'height' and $x$ and $y$ to be 'Eastings' and 'Northings', the graph of the function $F$ corresponds to the hillside of our intuitive discussion.

Since $z = \sqrt{1 - (x^2 + y^2)}$, it follows that $x^2 + y^2 + z^2 = 1$ so that all points of the graph lie on the surface of a unit sphere with centre at the origin.

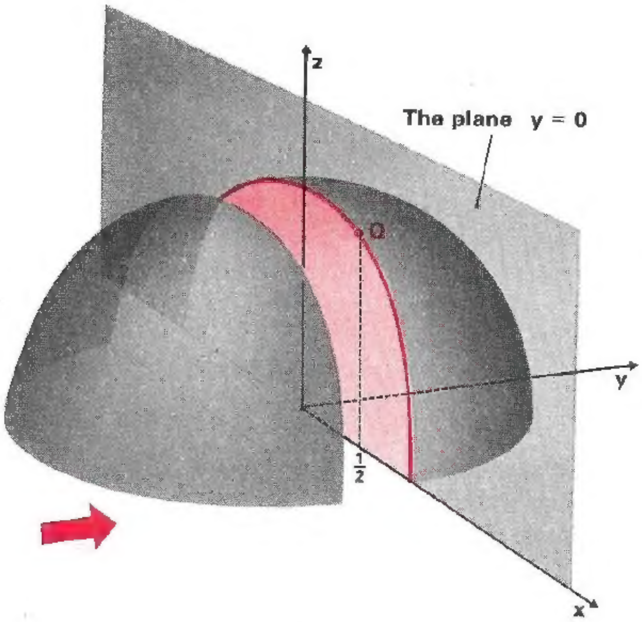
Choosing $z$ to be always positive means that its graph is in fact a hemisphere, looking a little like a hill.

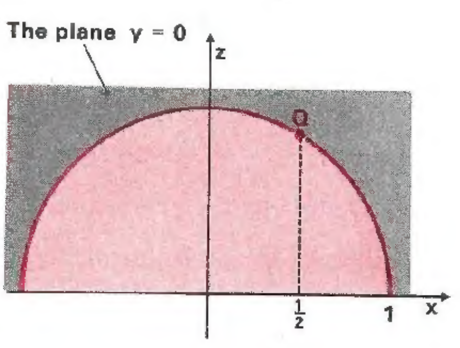
On this hill we choose, for illustration, the point Q with co-ordinates $\left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)$ and we take two roads through Q given by the intersection of the hemisphere with the vertical planes through Q parallel to the $x$ and $y$ axes.




The whole of the east–west road lies in the plane $y = 0$.

If we were to cut the hemisphere with the plane $y = 0$ through Q, and then look along the $y$-axis, we would see the semi-circle shown in red in the following diagram:




The semi-circle is the intersection of the plane $y = 0$ with the hemisphere $z = \sqrt{1 - (x^2 + y^2)}$ and so its equation, in the plane $y = 0$, is

$$z = \sqrt{1 - x^2}, \qquad x \in [-1, +1].$$

Defining a function $f_1$ by

$$f_1(x) = \sqrt{1 - x^2}, \qquad (x \in [-1, +1]$$

we can calculate the slope of the road at Q from the derivative

$$f_1'(x) = \frac{-x}{\sqrt{1-x^2}}.$$

When $x = \frac{1}{2}$, this takes the value $-\frac{1}{\sqrt{3}}$, which is the slope of the easterly road through Q.

The north–south road similarly lies at the intersection of the plane $x = \frac{1}{2}$ with the hemisphere $z = \sqrt{1-(x^2+y^2)}$.

We thus have

$$z = \sqrt{1-(\tfrac{1}{4}+y^2)} = \sqrt{\tfrac{3}{4}-y^2}, \qquad \left(y \in \left[-\frac{\sqrt{3}}{2}, +\frac{\sqrt{3}}{2}\right]\right)$$

Defining a function $f_2$ by

$$f_2(y) = \sqrt{\tfrac{3}{4}-y^2}, \qquad \left(y \in \left[-\frac{\sqrt{3}}{2}, +\frac{\sqrt{3}}{2}\right]\right),$$

we calculate the slope from the derivative

$$f_2'(y) = \frac{-y}{\sqrt{\frac{3}{4}-y^2}}.$$

When $y = 0$ this takes the value 0, which is the slope of the northerly road through Q.

### 2.2.2 Definition of Partial Derivatives

The intuitive idea of the previous example points the way to a more general definition.

Consider a point $P = (a, b, c)$ on the hemisphere, so that $c \geq 0$ and $a^2 + b^2 + c^2 = 1$. Through P we have the two planes $y = b$ and $x = a$ and we may seek the slope at P of the semi-circles in which each plane meets the hemisphere.

Keeping $y$ fixed at the constant value $b$, we obtain a function of just one variable:

$$f_1(x) = \sqrt{1-(x^2+b^2)}, \qquad x \in [-\sqrt{1-b^2}, +\sqrt{1-b^2}]$$

This function has a graph which is a semi-circle in the plane $y = b$ and the slope is given by the derivative

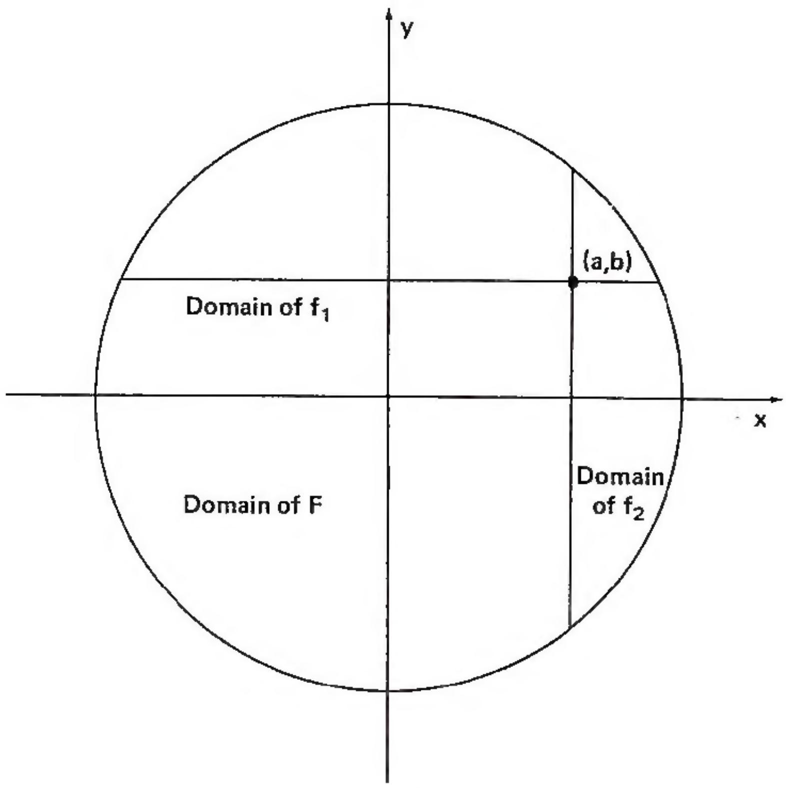
$$f_1'(x) = \frac{-x}{\sqrt{1-(x^2+b^2)}}.$$

Similarly, if we keep $x$ fixed at the constant value $a$, we get another function of a single variable:

$$f_2(x) = \sqrt{1-(a^2+y^2)}, \qquad y \in [-\sqrt{1-a^2}, +\sqrt{1-a^2}].$$

The graph of this function is a semicircle in the plane $x = a$ and its slope is given by the derivative

$$f_2'(y) = \frac{-y}{\sqrt{1-(a^2+y^2)}}.$$




By defining $f_1'(x)$ for each value of $b \in [-1, +1]$, we may construct a new function of *two* variables:

$$F_1'(x, y) = \frac{-x}{\sqrt{1-(x^2+y^2)}}, \qquad (x, y) \in \{(x, y) : x^2 + y^2 < 1\}$$

Likewise, defining a function $f_2'(y)$ for each $a \in [-1, +1]$, we construct another function of two variables:

$$F_2'(x, y) = \frac{-y}{\sqrt{1-(x^2+y^2)}}, \qquad (x, y) \in \{(x, y) : x^2 + y^2 < 1\}.$$

$F_1'$ and $F_2$ are the two *partial* derived functions of $F$.

**Exercise**

Calculate $F_1'(\frac{1}{2}, 0)$ and $F_2'(\frac{1}{2}, 0)$ and compare your answers with the slopes obtained in the example of the previous section.

*Solution*

$$F_1'(\tfrac{1}{2}, 0) = \frac{-\frac{1}{2}}{\sqrt{1-((\frac{1}{2})^2+0^2)}} = \frac{-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} = -\frac{1}{\sqrt{3}}.$$

$$F_2'(\tfrac{1}{2}, 0) = \frac{-0}{\sqrt{1-((\frac{1}{2})^2+0^2)}} = \frac{0}{\sqrt{\frac{3}{4}}} = 0.$$

These values agree with those previously obtained for the slopes at Q.

Remembering that all we do is to keep each variable in turn fixed whilst we differentiate with respect to the other variable, we can define the *partial derivatives* of a general function

$$F : R \times R \to R.$$

The *partial derivative* of $F$ with respect to the *first* variable, $x$, at the point $(x, y)$ is

$$F_1'(x, y) = \lim_{h \to 0} \frac{F(x+h, y) - F(x, y)}{h}.$$

The *partial derivative* of $F$ with respect to the *second* variable, $y$, at the point $(x, y)$ is

$$F_2'(x, y) = \lim_{k \to 0} \frac{F(x, y + k) - F(x, y)}{k}.$$

*Example 1*

Consider the function

$$G: (x, y) \longmapsto 2xy + x^2, \qquad (x, y) \in R \times R.$$

Treating $y$ as a constant, we may differentiate with respect to $x$ and get

$$G_1'(x, y) = 2y + 2x.$$

Similarly, treating $x$ as a constant, we may differentiate with respect to $y$ and get

$$G_2'(x, y) = 2x.$$

**Exercise**

Verify the results of the above example by working directly from the limit definitions.

*Solution*

$$\begin{aligned} G_1'(x, y) &= \lim_{h \to 0} \left( \frac{2y(x + h) + (x + h)^2 - (2xy + x^2)}{h} \right) \\ &= \lim_{h \to 0} \left( \frac{2yh + 2xh + h^2}{h} \right) \\ &= 2y + 2x. \\ G_2'(x, y) &= \lim_{k \to 0} \left( \frac{2x(y + k) + x^2 - (2xy + x^2)}{k} \right) \\ &= \lim_{k \to 0} \left( \frac{2xk}{k} \right) \\ &= 2x. \end{aligned}$$

**Exercise**

Each of the functions defined below has domain $R \times R$. Find all their partial derivatives.

(i) $F(x, y) = x^2 + y^2$.

(ii) $G(x, y) = x \exp(xy)$.

(iii) $H(x, y) = x \sin(x + y)$.

(iv) $P(x, y) = x^4 + y^4 - 4x^2y^3$.

*Solution*

(i) $F_1'(x, y) = 2x$,
$F_2'(x, y) = 2y$.

(ii) $G_1'(x, y) = \exp(xy) + xy \exp(xy)$,
$G_2'(x, y) = x^2 \exp(xy)$

(iii) $H_1'(x, y) = \sin(x + y) + x \cos(x + y)$,
$H_2'(x, y) = x \cos(x + y)$.

(iv) $P_1'(x, y) = 4x^3 - 8xy^3$,
$P_2'(x, y) = 4y^3 - 12x^2y^2$.

## Notation

You will meet various alternative notations for partial derivatives.

Thus $F_x$ is often used where we have $F_1'$, and $F_y$ for $F_2'$.

The commonest notation of all is

$$\frac{\partial F}{\partial x} \text{ for } F_1'(x, y) \text{ and}$$

$$\frac{\partial F}{\partial y} \text{ for } F_2'(x, y).$$

This is reminiscent of the use of $\dfrac{df}{dt}$ for the ordinary derivative $f'(t)$. However, be very wary of jumping to conclusions: it is *not* generally true that $\dfrac{\partial F}{\partial x} \cdot \dfrac{\partial x}{\partial t}$ is the same as $\dfrac{\partial F}{\partial t}$, whatever the notation may suggest.